

A generalized Omori-Yau maximum principle and Liouville-type theorems for a second-order linear semi-elliptic operator

Kyusik Hong and Chanyoung Sung

Department of Mathematics, Konkuk University, Seoul, 143-701, Republic of Korea.

Abstract. We generalize the Omori-Yau almost maximum principle of the Laplace-Beltrami operator on a complete Riemannian manifold M to a second-order linear semi-elliptic operator $\tilde{\Delta}$ with bounded coefficients and no zeroth order term.

Using this result, we prove some Liouville-type theorems for a real-valued C^2 function f on M satisfying $\tilde{\Delta}f \geq F(f) + H(|\nabla f|)$ for real-valued continuous functions F and H on \mathbb{R} such that $H(0) = 0$.

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1. INTRODUCTION

Let (M, g) be a smooth complete Riemannian manifold of dimension n . A second-order linear differential operator $\tilde{\Delta} : C^\infty(M) \rightarrow C^\infty(M)$ without zeroth order term can be written as

$$\tilde{\Delta}(f) = \text{Tr}(A \circ \text{hess}(f)) + g(V, \nabla f),$$

where $A \in \Gamma(\text{End}(TM))$ is self-adjoint with respect to g , $\text{hess}(f) \in \Gamma(\text{End}(TM))$ is the Hessian of f in the form defined by $\text{hess}(f)(X) = \nabla_X \nabla f$ for $X \in \Gamma(TM)$, and finally $V \in \Gamma(TM)$. In this article, we will deal with the semi-elliptic case, i.e. A is positive semi-definite at each point, and we always assume that

$$\sup_M \text{Tr}(A) + \sup_M \|V\| < \infty.$$

The purpose of this paper is that such a operator D shares important properties with the Laplace-Beltrami operator, particularly Omori-Yau almost maximum principle and Liouville-type theorems for subharmonic functions.

To state our main theorem, we need the following definitions.

Definition 1.1. Let u be a real-valued continuous function on M and let a point $p \in M$.

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²E-mail addresses : kszoo@postech.ac.kr, cysung@kias.re.kr

- a function u is called proper, if the set $\{p : u(p) \leq r\}$ is compact for every real number r .
- a function v defined on a neighborhood U_p of p is called an upper-supporting function for u at p , if the conditions $v(p) = u(p)$ and $v \geq u$ hold in U_p .

Definition 1.2. A proper function $u : M \rightarrow \mathbb{R}$ is called a tamed-exhaustion, if the following condition holds:

- (1) $u \geq 0$,
- (2) At all points $p \in M$ it has a C^2 smooth, upper-supporting function v at p defined on an open neighborhood U_p such that $\|\nabla v\| \leq 1$ and $\Delta v \leq 1$ at p , where Δ is the Laplace-Beltrami operator.

The existence of a tamed exhaustion function on a complete Riemannian manifold is not quite a restrictive condition. In [3], it is shown that such a function exists if the Ricci curvature Ric satisfies

$$\text{Ric}(\nabla r, \nabla r) \geq -B\rho(r)$$

for some constant $B > 0$, where r is the distance from an arbitrarily fixed point in M and a smooth nondecreasing function $\rho(r)$ on $[0, \infty)$ satisfies

$$\begin{aligned} \rho(0) = 1, \quad \rho^{(2k+1)}(0) = 0 \quad \forall k \geq 0, \\ \sqrt{\rho(t)} \notin L^1, \quad \limsup_{t \rightarrow \infty} \frac{t\rho(\sqrt{t})}{\rho(t)} < \infty. \end{aligned}$$

For example, if

$$\text{Ric}(\nabla r, \nabla r) \geq -B'r^2(\log r)^2(\log(\log r))^2 \cdots (\log^k r)^2$$

for $r \gg 1$, a tamed exhaustion always exists.

We now present our main results.

Theorem 1.3. *Let M be a smooth complete Riemannian manifold admitting a tamed exhaustion function. Suppose that a C^2 function $f : M \rightarrow \mathbb{R}$ is bounded below and satisfies $\tilde{\Delta}f \geq F(f) + H(|\nabla f|)$ for real-valued continuous functions F and H on \mathbb{R} such that $H(0) = 0$.*

- (1) *If $\liminf_{x \rightarrow \infty} \frac{F(x)}{x^\nu} > 0$ for some $\nu > 1$, then f is bounded such that $F(\sup f) \leq 0$.*
- (2) *If $\liminf_{x \rightarrow \infty} \frac{F(x)}{x^\nu} \leq 0$ for any $\nu > 1$, then $\sup f = \infty$ or f is bounded such that $F(\sup f) \leq 0$.*

Theorem 1.4. *Let M be as in theorem 1.3. Suppose that a C^2 function $f : M \rightarrow \mathbb{R}$ is bounded above and satisfies $\tilde{\Delta}f \geq F(f) + H(|\nabla f|)$ for F and H as in the above theorem.*

- (1) *If $\liminf_{x \rightarrow -\infty} \frac{F(x)}{(-x)^\nu} > 0$ for some $\nu > 1$, then f is bounded such that $F(\inf f) \leq 0$.*
- (2) *If $\liminf_{x \rightarrow -\infty} \frac{F(x)}{(-x)^\nu} \leq 0$ for any $\nu > 1$, then $\inf f = -\infty$ or f is bounded such that $F(\inf f) \leq 0$.*

As a corollary, we give a semi-elliptic generalization of Liouville's theorem stating that any $f \in C^2(\mathbb{R}^2)$ which is subharmonic ($\Delta f \geq 0$) and bounded above must be constant.

Corollary 1.5. *Let M be as in theorem 1.3.*

- (1) *There exists no $f \in C^2(M)$ which is bounded above and $\tilde{\Delta}f \geq c$ for a constant $c > 0$.*
- (2) *Any $f \in C^2(M)$ which is non-positive and satisfies $\tilde{\Delta}f \geq c|f|^d$ for some positive constants c and d must be identically zero.*

The proofs of theorems are based on the Omori-Yau almost maximum principle. In section 2, inspired by the article [3] of K.-T. Kim and H. Lee, we generalize the Omori-Yau maximum principle of the Laplace-Beltrami operator to the above semi-elliptic operator $\tilde{\Delta}$.

Remark 1.6. Theorem 1.3, 1.4 and corollary 1.5 can be easily extended to any second-order linear semi-elliptic operator

$$\tilde{\Delta} + h,$$

for $h \in C^\infty(M)$ just by considering $\tilde{\Delta}f \geq F(f) - hf + H(\nabla f)$.

2. GENERALIZED OMORI-YAU MAXIMUM PRINCIPLE

Theorem 2.1. *Let M be a smooth complete Riemannian manifold admitting a tamed exhaustion function. Then for every real-valued C^2 function f on M is bounded above, there exists a sequence $\{p_k\}$ on M satisfying the following properties:*

$$\lim_{k \rightarrow \infty} \|\nabla f\|(p_k) = 0, \quad \limsup_{k \rightarrow \infty} \tilde{\Delta}f(p_k) \leq 0, \quad \text{and} \quad \lim_{k \rightarrow \infty} f(p_k) = \sup_M f.$$

Proof. The proof is similar to the method in the article [3]. Without loss of generality, we may assume that $\sup_M f > 0$ by adding some positive constant.

Now, we choose a point $p \in M$ such that $f(p) > 0$. For each $\epsilon > 0$, let

$$X_\epsilon = \{x \in M \mid u(x) < \frac{1}{\epsilon}\}.$$

Then X_ϵ forms an increasing sequence of open subsets of M and each closure \overline{X}_ϵ gives rise to a compact exhaustion of M as $\epsilon \downarrow 0$.

Taking a positive constant r such that $p \in X_r$. Consider the continuous function

$$(1 - ru(x))f(x).$$

Because $(1 - ru(p))f(p) > 0$ and $(1 - ru(x))f(x)$ vanishes on the boundary of X_r . Moreover, because \overline{X}_r is compact. The function $(1 - ru)f$ attains its maximum value in the set X_r , say at $p_r \in X_r$, respectively. It is obvious that the maximum value is positive. From now on, we fix r .

Let ϵ be any positive constant smaller than r . Then $p \in X_r \subset X_\epsilon$ and

$$(1 - \epsilon u(p))f(p) \geq (1 - ru(p))f(p) > 0.$$

The function $(1 - \epsilon u)f$ attains a positive maximum value in the set X_ϵ , say at $p_\epsilon \in X_\epsilon$.

Since A is symmetric, it is diagonalizable at each point in an orthonormal basis, so we can take a normal coordinate (x_1, \dots, x_n) around $p_\epsilon \in M$ such that A at p_ϵ is represented as a diagonal matrix, and hence

$$(2.2) \quad \tilde{\Delta}u|_{p_\epsilon} = \sum_l a_{ll}(p_\epsilon) \frac{\partial^2}{\partial x_l^2} u|_{p_\epsilon} + \sum_l a_l(p_\epsilon) \frac{\partial}{\partial x_l} u|_{p_\epsilon},$$

where each $a_{ll}(p_\epsilon)$ is nonnegative, and the entries $a_{ll}(p_\epsilon)$ and $|a_l(p_\epsilon)|$ are bounded above as p_ϵ varies. For a notational convenience, let's introduce locally-defined differential operators

$$(2.3) \quad \tilde{\nabla} := a_{11}(p_\epsilon) \frac{\partial}{\partial x_1} + \dots + a_{nn}(p_\epsilon) \frac{\partial}{\partial x_n} \text{ and } \tilde{\nabla}_1 := a_1(p_\epsilon) \frac{\partial}{\partial x_1} + \dots + a_n(p_\epsilon) \frac{\partial}{\partial x_n}.$$

If u has an extremal value at point p_ϵ ,

$$\tilde{\Delta}u|_{p_\epsilon} = \sum_l a_{ll}(p_\epsilon) \frac{\partial^2}{\partial x_l^2} u|_{p_\epsilon}.$$

Put $d_l = a_{ll}(p_\epsilon)$ for $1 \leq l \leq n$.

Furthermore, if a real-valued function AB on M has an extremal value at p_ϵ , then one can obtain

$$(2.4) \quad \tilde{\Delta}(AB)|_{p_\epsilon} = (\tilde{\Delta}A|_{p_\epsilon} - \tilde{\nabla}_1 A|_{p_\epsilon})B(p_\epsilon) + 2\tilde{\nabla}A|_{p_\epsilon} \cdot \nabla B|_{p_\epsilon} + A(p_\epsilon)(\tilde{\Delta}B|_{p_\epsilon} - \tilde{\nabla}_1 B|_{p_\epsilon}).$$

Note that $2(\tilde{\nabla}A|_{p_\epsilon} \cdot \nabla B|_{p_\epsilon}) = 2(\nabla A|_{p_\epsilon} \cdot \tilde{\nabla}B|_{p_\epsilon})$. Put $e_l = |a_l(p_\epsilon)|$ for $1 \leq l \leq n$.

We may assume that d_1 and e_1 are the largest of $\{d_1, \dots, d_n\}$ and $\{e_1, \dots, e_n\}$ respectively. Consider a C^2 upper-supporting function $v : U \rightarrow \mathbb{R}$ for u at p_ϵ , where U is an open neighborhood of p_ϵ . Then we get

$$\|\tilde{\nabla}v|_{p_\epsilon}\| \leq d_1, \quad \|\tilde{\nabla}_1 v|_{p_\epsilon}\| \leq e_1, \quad \text{and } \tilde{\Delta}v|_{p_\epsilon} \leq d_1 + e_1.$$

By taking U further small, we may assume that $U \subset X_\epsilon$ and f is positive on U , since $f(p_\epsilon) > 0$. For every $x \in U$,

$$(1 - \epsilon v(x))f(x) \leq (1 - \epsilon u(x))f(x) \leq (1 - \epsilon u(p_\epsilon))f(p_\epsilon) = (1 - \epsilon v(p_\epsilon))f(p_\epsilon).$$

Since p_ϵ is a local maximum point of $(1 - \epsilon v)f$, we get

$$\nabla[(1 - \epsilon v)f]|_{p_\epsilon} = \tilde{\nabla}[(1 - \epsilon v)f]|_{p_\epsilon} = \tilde{\nabla}_1[(1 - \epsilon v)f]|_{p_\epsilon} = 0.$$

By a simple calculation, we have

$$(1 - \epsilon v(p_\epsilon))\|\tilde{\nabla}f(p_\epsilon)\| = \epsilon\|\tilde{\nabla}v(p_\epsilon)\|f(p_\epsilon) \leq \epsilon d_1(\sup_M f).$$

From $v(p_\epsilon) = u(p_\epsilon)$, we get

$$(1 - \epsilon u(p_\epsilon))\|\tilde{\nabla}f(p_\epsilon)\| \leq \epsilon d_1(\sup_M f).$$

Also, because $X_r \subset X_\epsilon$. We have

$$(1 - \epsilon u(p_r))f(p_r) \leq (1 - \epsilon u(p_\epsilon))f(p_\epsilon).$$

This implies that

$$\begin{aligned} (1 - ru(p_r))f(p_r)\|\tilde{\nabla}f(p_\epsilon)\| &\leq (1 - \epsilon u(p_r))f(p_r)\|\tilde{\nabla}f(p_\epsilon)\| \leq (1 - \epsilon u(p_\epsilon))f(p_\epsilon)\|\tilde{\nabla}f(p_\epsilon)\| \\ &\leq f(p_\epsilon)\epsilon d_1(\sup_M f) \leq \epsilon d_1(\sup_M f)^2. \end{aligned}$$

So, we conclude that

$$\|\tilde{\nabla}f(p_\epsilon)\| \leq \epsilon \frac{d_1(\sup_M f)^2}{(1 - ru(p_r))f(p_r)}.$$

Note that $K := \frac{d_1(\sup_M f)^2}{(1 - ru(p_r))f(p_r)}$ is a positive constant independent of ϵ with $\epsilon < r$. Therefore, we obtain

$$\lim_{\epsilon \rightarrow 0} \|\tilde{\nabla}f(p_\epsilon)\| = 0.$$

By the same method as above, we have

$$\|\nabla f(p_\epsilon)\| \leq \epsilon \frac{(\sup_M f)^2}{(1 - ru(p_r))f(p_r)} \quad \text{and} \quad \|\tilde{\nabla}_1 f(p_\epsilon)\| \leq \epsilon \frac{e_1(\sup_M f)^2}{(1 - ru(p_r))f(p_r)}$$

Therefore, we get

$$\lim_{\epsilon \rightarrow 0} \|\nabla f(p_\epsilon)\| = 0 \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} \|\tilde{\nabla}_1 f(p_\epsilon)\| = 0.$$

Now we prove

$$\limsup_{\epsilon \rightarrow 0} \tilde{\Delta}f(p_\epsilon) \leq 0.$$

Since p_ϵ is a local maximum point of $(1 - \epsilon v)f$, we have $\tilde{\Delta}((1 - \epsilon v)f) \leq 0$ at point p_ϵ . Using the formula (2.4),

$$\begin{aligned} [\tilde{\Delta}((1 - \epsilon v)f)]|_{p_\epsilon} &= -\epsilon \tilde{\Delta}v|_{p_\epsilon} f(p_\epsilon) + \epsilon f(p_\epsilon) \tilde{\nabla}_1 v|_{p_\epsilon} - 2\epsilon \nabla v|_{p_\epsilon} \cdot \tilde{\nabla}f|_{p_\epsilon} + (1 - \epsilon v(p_\epsilon)) \tilde{\Delta}f|_{p_\epsilon} \\ &\quad - (1 - \epsilon v(p_\epsilon)) \tilde{\nabla}_1 f|_{p_\epsilon} \\ &\leq 0. \end{aligned}$$

Hence

$$\begin{aligned} (1 - \epsilon v(p_\epsilon)) \tilde{\Delta}f|_{p_\epsilon} &\leq 2\epsilon \nabla v|_{p_\epsilon} \cdot \tilde{\nabla}f|_{p_\epsilon} + \epsilon \tilde{\Delta}v|_{p_\epsilon} f(p_\epsilon) - \epsilon f(p_\epsilon) \tilde{\nabla}_1 v|_{p_\epsilon} + (1 - \epsilon v(p_\epsilon)) \tilde{\nabla}_1 f|_{p_\epsilon} \\ &\leq \epsilon (2\|\tilde{\nabla}f|_{p_\epsilon}\| + (d_1 + e_1) \sup_M f + e_1 \sup_M f) + \|\tilde{\nabla}_1 f|_{p_\epsilon}\| (1 - \epsilon v(p_\epsilon)) \\ &\leq \epsilon (2\epsilon K + (d_1 + e_1) \sup_M f + e_1 \sup_M f) + \|\tilde{\nabla}_1 f|_{p_\epsilon}\| (1 - \epsilon v(p_\epsilon)). \end{aligned}$$

Since $(1 - \epsilon u(p_\epsilon)) = (1 - \epsilon v(p_\epsilon)) > 0$, we get

$$\tilde{\Delta}f|_{p_\epsilon} \leq \epsilon \frac{(2\epsilon K + (d_1 + e_1) \sup_M f + e_1 \sup_M f)}{(1 - \epsilon u(p_\epsilon))} + \|\tilde{\nabla}_1 f|_{p_\epsilon}\|.$$

As above, we obtain

$$\begin{aligned} \tilde{\Delta}f|_{p_\epsilon} &\leq \epsilon \frac{(2\epsilon K + (d_1 + e_1) \sup_M f + e_1 \sup_M f)(\sup_M f)}{(1 - \epsilon u(p_\epsilon))f(p_\epsilon)} + \epsilon \frac{e_1(\sup_M f)^2}{(1 - ru(p_r))f(p_r)} \\ &\leq \epsilon \frac{(2\epsilon K + (d_1 + e_1) \sup_M f + e_1 \sup_M f)(\sup_M f)}{(1 - ru(p_r))f(p_r)} + \epsilon \frac{e_1(\sup_M f)^2}{(1 - ru(p_r))f(p_r)}. \end{aligned}$$

Therefore, we conclude that there is a positive constant C independent of ϵ such that $\tilde{\Delta}f|_{p_\epsilon} \leq C\epsilon$.

It only remains to show that $\lim_{\epsilon \rightarrow 0} f(p_\epsilon) = \sup_M f$.

Let η be any positive constant such that $\sup_M f > \eta$. We may choose a point $q \in M$ such that $f(q) > \sup_M f - \frac{\eta}{2}$, because f is bounded above. Also, we choose a positive constant ϵ with $\epsilon < r$ such that $q \in X_\epsilon$. Taking a sufficiently small value for ϵ , we get

$$(1 - \epsilon u(p_\epsilon))f(p_\epsilon) \geq (1 - \epsilon u(q))f(q) \geq \sup_M f - \eta.$$

Since $0 < 1 - \epsilon u(p_\epsilon) < 1$, we have

$$f(p_\epsilon) \geq \frac{\sup_M f - \eta}{(1 - \epsilon u(p_\epsilon))} > \sup_M f - \eta.$$

□

Remark 2.5. Recently L.J. Alias, D. Impera, and M. Rigoli [1] have proved a generalized Omori-Yau maximum principle as above under a stronger assumption.

They used it to obtain certain estimates of higher order mean curvatures of hypersurfaces in some warped product spaces, and D. Impera [2] similarly obtained such estimates for spacelike hypersurfaces in Lorentzian manifolds.

3. PROOF OF THEOREM 1.3

We follow the idea of [4]. We may choose a constant a such that $f + a > 0$, because f is bounded below. Let $G : M \rightarrow \mathbb{R}^+$ be a C^2 function such that $G = (f + a)^{\frac{1-q}{2}}$ where $q > 1$ is a constant.

Since G is bounded below, the proof of theorem 2.1 tells us that for any $\delta > 0$ there exists a point $p_\epsilon \in M$ such that

$$(3.1) \quad |\nabla G(p_\epsilon)| < \delta, \quad |\tilde{\nabla} G(p_\epsilon)| < \delta, \quad \tilde{\Delta} G(p_\epsilon) > -\delta, \quad \text{and} \quad \inf G + \delta > G(p_\epsilon),$$

where $\tilde{\nabla}$ is defined by (2.3). Note that $G(p_\epsilon) \rightarrow \inf G$ and $f(p_\epsilon) \rightarrow \sup f$ as $\delta \rightarrow 0$.

By a direct calculation,

$$(3.2) \quad \tilde{\nabla} G|_{p_\epsilon} = \left(\frac{1-q}{2}\right)G(p_\epsilon)^{\frac{q+1}{q-1}}\tilde{\nabla} f|_{p_\epsilon}.$$

Lemma 3.3.

$$(3.4) \quad \tilde{\Delta} G|_{p_\epsilon} = -\left(\frac{q+1}{2}\right)G(p_\epsilon)^{\frac{2}{q-1}}\nabla G|_{p_\epsilon} \cdot \tilde{\nabla} f|_{p_\epsilon} + \left(\frac{1-q}{2}\right)G(p_\epsilon)^{\frac{q+1}{q-1}}\tilde{\Delta} f|_{p_\epsilon}.$$

Proof. By (2.2), evaluating $\tilde{\Delta} G$ at p_ϵ , we have

$$\tilde{\Delta} G|_{p_\epsilon} = \sum_l a_l(p_\epsilon) \frac{\partial^2}{\partial x_l^2} G|_{p_\epsilon} + \sum_l a_l(p_\epsilon) \frac{\partial}{\partial x_l} G|_{p_\epsilon}, \quad \text{where } 1 \leq l \leq n.$$

By a simple calculation, one gets

$$\sum_l a_l(p_\epsilon) \frac{\partial^2}{\partial x_l^2} G|_{p_\epsilon} = -\left(\frac{q+1}{2}\right)G(p_\epsilon)^{\frac{2}{q-1}}\nabla G|_{p_\epsilon} \cdot \tilde{\nabla} f|_{p_\epsilon} + \sum_l a_l(p_\epsilon) \left(\frac{1-q}{2}\right)(f+a)^{\frac{-1-q}{2}} \frac{\partial^2}{\partial x_l^2} (f+a)$$

and

$$\sum_l a_l(p_\epsilon) \frac{\partial}{\partial x_l} G|_{p_\epsilon} = \sum_l a_l(p_\epsilon) \left(\frac{1-q}{2}\right) (f+a)^{\frac{-1-q}{2}} \frac{\partial}{\partial x_l} (f+a).$$

This yields a desired equality. \square

By plugging (3.2) to (3.4), we have

$$\left(\frac{1-q}{2}\right) G(p_\epsilon)^{\frac{2q}{q-1}} \tilde{\Delta} f|_{p_\epsilon} = G(p_\epsilon) \tilde{\Delta} G|_{p_\epsilon} - \left(\frac{q+1}{q-1}\right) \nabla G(p_\epsilon) \cdot \tilde{\nabla} G(p_\epsilon).$$

By (3.1),

$$(3.5) \quad \left(\frac{1-q}{2}\right) G(p_\epsilon)^{\frac{2q}{q-1}} \tilde{\Delta} f|_{p_\epsilon} > G(p_\epsilon)(-\delta) - \left(\frac{q+1}{q-1}\right) \delta^2.$$

Applying $\tilde{\Delta} f \geq F(f) + H(|\nabla f|)$ and replacing G by $(f+a)^{\frac{1-q}{2}}$, we have

$$(3.6) \quad \frac{F(f(p_\epsilon)) + H(|\nabla f(p_\epsilon)|)}{(f(p_\epsilon) + a)^q} < \left(\frac{2\delta}{q-1}\right) \frac{1}{(f(p_\epsilon) + a)^{\frac{q-1}{2}}} + \frac{2(q+1)}{(q-1)^2} \delta^2.$$

Assume that $\sup f < \infty$. Then as $\delta \rightarrow 0$, since $\nabla G|_{p_\epsilon} \rightarrow 0$, G is bounded below by positive constant, and

$$\nabla G|_{p_\epsilon} = \left(\frac{1-q}{2}\right) G(p_\epsilon)^{\frac{q+1}{q-1}} \nabla f|_{p_\epsilon},$$

we have $H(|\nabla f(p_\epsilon)|) \rightarrow 0$. Also, the **RHS** of (3.6) converges to 0 while the **LHS** of (3.6) converges to $\frac{F(\sup f)}{(\sup f + a)^q}$ as $\delta \rightarrow 0$. Thus, we get $F(\sup f) \leq 0$.

Now, it is enough to show that when $\liminf_{x \rightarrow \infty} \frac{F(x)}{x^\nu} > 0$ for some $\nu > 1$, f must be bounded. Suppose that $\sup f = \infty$. Then for $q < \nu$, the **RHS** of (3.6) converges to 0, while the **LHS** of (3.6) diverges to ∞ as $\delta \rightarrow 0$. This is a contradiction. Therefore, f must be bounded.

4. PROOF OF THEOREM 1.4

We follow the idea of [4]. Since $-f$ is bounded below, we can apply the proof of theorem 1.3 to $-f$ with $q < 1$. By the inequality (3.5), we get

$$\left(\frac{1-q}{2}\right) G(p_\epsilon)^{\frac{2q}{q-1}} \tilde{\Delta}(-f)|_{p_\epsilon} > G(p_\epsilon)(-\delta) - \frac{|q+1|}{|q-1|} \delta^2.$$

Applying $\tilde{\Delta} f \geq F(f) + H(|\nabla f|)$, we have

$$\frac{F(f(p_\epsilon)) + H(|\nabla f(p_\epsilon)|)}{(-f(p_\epsilon) + a)^q} \leq \frac{\tilde{\Delta} f(p_\epsilon)}{(-f(p_\epsilon) + a)^q} < \left(\frac{2\delta}{1-q}\right) \frac{1}{(-f(p_\epsilon) + a)^{\frac{q-1}{2}}} + \frac{2|q+1|}{(q-1)^2} \delta^2.$$

By a simple calculation,

$$(4.1) \quad \frac{F(f(p_\epsilon)) + H(|\nabla f(p_\epsilon)|)}{(-f(p_\epsilon) + a)^{\frac{q+1}{2}}} < \frac{2\delta}{1-q} + \frac{2|q+1|}{(q-1)^2} \delta^2 (-f(p_\epsilon) + a)^{\frac{q-1}{2}}.$$

By the same method as above, we get $H(|\nabla f(p_\epsilon)|) \rightarrow 0$. If $\inf f > -\infty$, then $F(\inf f) \leq 0$ as $\delta \rightarrow 0$.

Now it only remain to show that if $\liminf_{x \rightarrow -\infty} \frac{F(x)}{(-x)^\nu} > 0$ for some $\nu < 1$, then f is bounded. Let's assume that to the contrary $\inf f = -\infty$. Then taking q such that $\frac{q+1}{2} < \nu$ and letting $\delta \rightarrow 0$. Then the **RHS** of (4.1) converges to 0 while the **LHS** of (4.1) diverges to ∞ . This is a contradiction completing the proof.

5. PROOF OF COROLLARY 1.5

Suppose that f is bounded above and satisfies $\tilde{\Delta}f \geq c > 0$ for a constant c . Applying Theorem 1.4 with $F = c$ and $H = 0$, one conclude that f is bounded and $F(\inf f) \leq 0$. This is contradictory to $F \equiv c > 0$.

For a proof of Corollary 1.5 (2), applying Theorem 1.4 with $F(f) = c|f|^d$, it follows that f is bounded and $c|\inf f|^d \leq 0$ implying $f \equiv 0$.

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